VECTOR CALCULUS
TOPIC III: CONICS

PAUL L. BAILEY

Abstract. We discuss four types of conics; the circle, the parabola, the ellipse, and the hyperbola. In each case, we derive an equation for the conic in standard position, and demonstrate a reflective principle for that conic.

1. Circles

Definition 1. A *circle* is the set of points in a plane whose distance to a given point, called the *center*, is equal to a given distance, called the *radius*.

Proposition 1. *Consider the circle centered at* $(0,0)$ *with radius* r*. The equation of the circle is*

$$\boxed{x^2 + y^2 = r^2.}$$

Proof. Let (x,y) be an arbitrary point on the circle; then its distance to the center is r. By the distance formula,

$$\sqrt{(x-0)^2 + (y-0)^2} = r,$$

so

$$x^2 + y^2 = r^2.$$

□

Date: September 8, 2017.

2. Parabolas

Definition 2. A *parabola* is the set of points in a plane whose distance to a given point, called the *focus*, is equal to its distance to a given line, called the *directrix*.

The midpoint between the focus and the directrix is called the *vertex*. The line through the focus and the vertex is called the *axis*.

Proposition 2. *Consider the parabola with vertex* $(0, 0)$ *and focus* $(0, p)$*. The axis is* $x = 0$*, the directrix is* $y = -p$*, and the equation the parabola is*

$$\boxed{x^2 = 4py.}$$

Proof. The line through $(0, 0)$ and $(0, p)$ is $x = 0$. The directrix is perpendicular to this, and the distance from $(0, p)$ to the directrix is $2p$. Thus the directrix is $y = -p$.

Let (x, y) be an arbitrary point on the parabola. Then the distance from (x, y) to $(0, p)$ equals the distance from (x, y) to the line $y = -p$. This latter distance is the distance between (x, y) and the point $(x, -p)$. Thus

$$\sqrt{(x-0)^2 + (y-p)^2} = \sqrt{(x-x)^2 + (y-(-p))^2}.$$

Squaring both sides gives

$$x^2 + (y-p)^2 = (y+p)^2.$$

Expanding gives

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2;$$

therefore

$$x^2 = 4py.$$

□

Proposition 3. *Consider a parabola with vertex V and focus F. Let A be the axis and let P be a point on the parabola. Let L_0 be the line through P tangent to the parabola, L_1 the line through P parallel to A, and L_2 the line through F and P. Then the angle between L_0 and L_1 equals the angle between L_0 and L_2.*

Proof. We may rotate and translate the parabola so that the focus is on the positive y-axis and the vertex is at the origin. Thus we assume that the focus is $F = (0, p)$ and the equation of the parabola is $4py = x^2$. The axis A is the line $x = 0$.

Viewing the parabola as the graph of a function, its equation in functional form is $f(x) = \frac{x^2}{4p}$. Calculus gives us that the slope of the line tangent to the graph of $f(x)$ at the point $(x, f(x))$ is $f'(x) = \frac{x}{2p}$.

Let $P = (a, b)$ be an arbitrary point on the parabola. Then $a^2 = 4bp$, and the slope of the tangent line L_0 at (a, b) is $\frac{a}{2p}$. The slope of the line L_2 through (a, b) and $(0, p)$ is $\frac{b-p}{a}$.

Let α be the angle between L_0 and L_1, and let β be the angle between L_0 and L_2. We wish to show that $\alpha = \beta$; since these angles are acute, it suffices to show that they have the same cosine. We use vectors to do this.

A vector in the direction of L_0 is $\vec{v} = \langle 2p, a\rangle$. A vector in the direction of L_1 is $\vec{j} = \langle 0, 1\rangle$. The cosine of the angle between them is

$$\cos\alpha = \frac{\vec{v}\cdot\vec{j}}{|\vec{v}||\vec{j}|} = \frac{a}{\sqrt{a^2+4p^2}}.$$

A vector in the direction of L_2 is $\vec{w} = \langle a, b-p\rangle$. The cosine of the angle between $\vec{v}$ and $\vec{w}$ is

$$\begin{aligned}\cos\beta &= \frac{\vec{v}\cdot\vec{w}}{|v||w|}\\ &= \frac{2ap+ab-ap}{\sqrt{a^2+4p^2}\sqrt{a^2+(p-b)^2}}\\ &= \frac{a(b+p)}{\sqrt{a^2+4p^2}\sqrt{4bp+(p-b)^2}}\\ &= \frac{a(b+p)}{\sqrt{a^2+4p^2}\sqrt{(p+b)^2}}\\ &= \frac{a}{\sqrt{a^2+4p^2}}\\ &= \cos\alpha\end{aligned}$$

Thus $\alpha = \beta$. □

Remark 1. This says that if a beam of light enters a circular paraboloid parallel to the axis, it will bounce off the surface and hit the focus.

3. Ellipses

Definition 3. An *ellipse* is the set of points in a plane such that the sum of the distances from the point to two given points, called *foci*, is a constant, called the *common sum*.

The midpoint between the foci is called the *center*. The line through the foci is called the *major axis*. The line perpendicular to the major axis through the center is called the *minor axis*. The points of intersection of the major axis with the ellipse are called *vertices*. The points of intersection of the minor axis with the ellipse are called *covertices*.

Proposition 4. *Consider the ellipse with foci $(\pm c, 0)$, where $c > 0$, and common sum s. Then the center is $(0,0)$, the major axis is $y = 0$, the minor axis is $x = 0$, the vertices are $(\pm a, 0)$, the covertices are $(0, \pm b)$, and the equation of the ellipse is*

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,}$$

where

$$\boxed{2a = s} \quad \text{and} \quad \boxed{c^2 = a^2 - b^2}.$$

Proof. The midpoint between the foci is clearly $(0,0)$, so this is the center. Moreover, the line through $(\pm c, 0)$ is the x-axis, so its equation is $y = 0$, and the perpendicular line through the origin is the y-axis, which is $x = 0$.

Suppose that the equation of the ellipse is as stated. If (x, y) is on the intersection of the locus of this equation with the line $y = 0$, then $\frac{x^2}{a^2} = 1$, so $x = \pm a$; thus the vertices are $(\pm a, 0)$. Similarly, the covertices are $(0, \pm b)$.

Now from the definition of an ellipse, the distance from $(a, 0)$ to $(c, 0)$ plus the distance from $(a, 0)$ to $(-c, 0)$ equals s, that is,

$$s = (a - c) + (a + c) = 2a.$$

Moreover, the distance from $(0, b)$ to $(c, 0)$ plus the distance from $(0, b)$ to $(-c, 0)$ equals s. Thus

$$s = \sqrt{(c-0)^2 + (0-b)^2} + \sqrt{(-c-0)^2 + (0-b)^2} = 2\sqrt{c^2 + b^2}.$$

Since $s = 2a$, this gives $a = \sqrt{c^2 + b^2}$, so $a^2 = c^2 + b^2$, which we rewrite as $c^2 = a^2 - b^2$. It remains to derive the equation of the ellipse from the definition.

Let (x, y) be an arbitrary point on the ellipse; from the definition, we have

$$\sqrt{(x-c)^2 + (y-0)^2} + \sqrt{(x-(-c))^2 + (y-0)^2} = s.$$

Subtracting $\sqrt{(x+c)^2 + y^2}$ from both sides and squaring gives

$$(x-c)^2 + y^2 = s^2 + (x+c)^2 + y^2 - 2s\sqrt{(x+c)^2 + y^2}.$$

Rearranging this gives

$$2s\sqrt{(x+c)^2 + y^2} = s^2 + (x+c)^2 - (x-c)^2 = s^2 + 4cx.$$

Dividing by $2s$ and squaring again produces

$$x^2 + 2cx + c^2 + y^2 = \frac{s^2}{4} + 2cx + \frac{4c^2x^2}{s^2}.$$

Cancelling $2cx$ and using that $s^2 = 4a^2$ and $c^2 = a^2 - b^2$ leads us to

$$x^2 + a^2 - b^2 + y^2 = a^2 + \frac{(a^2 - b^2)x^2}{a^2} = a^2 + x^2 - \frac{b^2x^2}{a^2}.$$

Adding $\frac{b^2x^2}{a^2} - x^2 - a^2 + b^2$ to both sides gives

$$\frac{b^2x^2}{a^2} + y^2 = b^2.$$

Finally, dividing by b^2 gives

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

□

Proposition 5. *Consider an ellipse with foci F_1 and F_2. Let P be a point on the ellipse and let L_0 be the line through P tangent to the ellipse. Let L_1 be the line through F_1 and P and let L_2 be the line through F_2 and P. Then the angle between L_0 and L_1 equals the angle between L_0 and L_2.*

Remark 2. This says that a wave emitted from one focus bounces off the surface and is transmitted to the other focus.

4. Hyperbolas

Definition 4. An *hyperbola* is the set of points in a plane such that the difference of the distances from the point to two given points, called *foci*, is a constant, called the *common difference*.

The midpoint between the foci is called the *center*. The line through the foci is called the *major axis*. The line perpendicular to the major axis through the center is called the *minor axis*. The points of intersection of the major axis with the hyperbola are called *vertices*. The points on the minor axis whose distance to the center is the square root of the difference of the squares of the distances from the center to the vertices and foci are called *covertices*.

Proposition 6. *Consider the hyperbola with foci* $(\pm c, 0)$*, where* $c > 0$*, and common difference* d*. Then the center is* $(0,0)$*, the major axis is* $y = 0$*, the minor axis is* $x = 0$*, the vertices are* $(\pm a, 0)$*, the covertices are* $(0, \pm b)$*, and the equation of the ellipse is*

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

where

$$\boxed{2a = d} \quad \text{and} \quad \boxed{c^2 = a^2 + b^2}.$$

Proposition 7. *Consider the hyperbola with equation*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The graph of this equation is asymptotic to the lines $y = \pm \frac{b}{a} x$.

Proof. Solving for y gives

$$y = \pm b \sqrt{\frac{x^2}{a^2} - 1} = \pm \frac{b}{a} \sqrt{x^2 - a^2}.$$

As $x \to \infty$, $\sqrt{x^2 - a^2} \to \sqrt{x^2} \to x$, so $y \to \pm \frac{b}{a} x$. □

Proposition 8. *Consider a hyperbola with foci* F_1 *and* F_2*. Let* P *be a point on the hyperbola and let* L_0 *be the line through* P *tangent to the hyperbola. Let* L_1 *be the line through* F_1 *and* P *and let* L_2 *be the line through* F_2 *and* P*. Then the angle between* L_0 *and* L_1 *equals the angle between* L_0 *and* L_2*.*

Remark 3. This is essentially the same statement we had for an ellipse, but because of the different shape of the hyperbola, we interpret it differently. It says that a wave emitted from a point outside the hyperbola directed at one focus bounces off the surface and is transmitted to the other focus.

5. Eccentricity

5.1. **Goal.** We wish to view all conic sections as a continuously varying family of curves, defined by a single equation which contains variables x and y, as well as *parameters*, which are constants which we allow to vary. That the family varies continuously means that small changes in the parameters cause small changes in the curves.

To simplify the ideas and the computations, we wish to fix the center of the conic sections to be the origin. Unfortunately, this precludes including parabolas in the family, and instead produces the degenerate case of two parallel lines where one would suspect a parabola to belong.

5.2. **A Family Parameterized by Time.** We begin by considering the equation

$$\frac{x^2}{a^2} + t\frac{y^2}{a^2} = 1.$$

For a fixed a and t, this is a conic section. With $|t| = \frac{a^2}{b^2}$, the standard equations of both ellipses and hyperbolas centered at the origin can be put in this form.

The parameter a represents the size of the curve, and we will now vary t and leave a fixed. Think of the parameter t as time, so that as time passes, the equation produces curves in a moving picture. All of the curves have x-intercepts at $(\pm a, 0)$.

For $t = 1$, the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1,$$

which is the equation of a circle of radius a centered at the origin.

For $t > 1$, as t increases, the curve flattens into an ellipse. For example, at $t = 4$, the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{(a/2)^2} = 1,$$

whose y-intercepts are $(0, \pm\frac{a}{2})$. As t approaches ∞, the curve approaches the horizontal line segment $[-a, a]$ on the x-axis.

For $0 < t < 1$, as t decreases from 1 to 0, the curve stretches vertically into an ellipse whose focal axis is the y-axis. For example, at $t = \frac{1}{4}$, the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{(2a)^2} = 1,$$

whose y-intercepts are $(0, \pm 2a)$. As t approaches 0, the ellipse becomes indefinitely tall.

For $t = 0$, the equation is

$$\frac{x^2}{a^2} = 1, \qquad \text{or} \qquad x = \pm a.$$

The graph of this equation is a pair of vertical lines.

For $t < 0$, the y^2 term is negative and we obtain the equation of a hyperbola. For example, at $t = -1$, the equation is

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1,$$

which is a hyperbola centered at the origin with horizontal focal axis and asymptotes $y = \pm x$. More generally, if $m > 0$ and $t = -m^2$, the asymptotes have slope $\frac{1}{m}$.

5.3. **Eccentricity.** The *eccentricity* of an ellipse or a hyperbola is

$$\boxed{e = \frac{c}{a} = \frac{\text{distance between foci}}{\text{distance between vertices}}.}$$

Thus $c = ae$.

For an ellipse, we can compute b^2 in terms of a and e as

$$c^2 = a^2 - b^2 \quad \Rightarrow \quad b^2 = a^2 - c^2 = a^2 - a^2e^2 = a^2(1 - e^2).$$

For a hyperbola, we have

$$c^2 = b^2 + a^2 \quad \Rightarrow \quad b^2 = c^2 - a^2 = a^2e^2 - a^2 = -a^2(1 - e^2),$$

so $-b^2 = a^2(1 - e^2)$.

In either case, the equation of the conic centered at the origin with size a and eccentricity e is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

5.4. **Generalized Parabolas.** Select a point P and line L not containing P. Let Q be another point on the plane, and let $d(Q, P)$ be the distance from Q to P, and let $d(Q, L)$ be the distance from Q to L. Then, by definition, Q is on the *parabola* with focus P and directrix L if and only if $d(Q, P) = d(Q, L)$. Dividing both sides by $d(Q, L)$ gives $\frac{d(Q,P)}{d(Q,L)} = 1$.

We generalize this as follows.

Select any point P, a line L not containing p, and a positive real number e. Consider the locus of the equation

$$\frac{d(Q, P)}{d(Q, L)} = e.$$

We call P the *focus*, L the *directrix*, and e the *eccentricity*, of this locus. If $e = 1$, this is a parabola. What type of locus does this equation create if $e \neq 1$? To understand this, let us change the scale and shift. We will rescale back at the end.

Let $a = \frac{ed(P,L)}{|e^2-1|}$; this is the *size* of the locus. Shrink the plane by a factor of a; for simplicity consider P, Q, and L in this shrunken plane without relabeling them. The distance between the focus and the directrix is now

$$d(P, L) = \frac{|e^2 - 1|}{e} = |e - \frac{1}{e}|.$$

By a rigid motion of the plane, assume that $P = (e, 0)$ and $L : x = \frac{1}{e}$. Then

$$\begin{aligned}
\frac{d(Q, p)}{d(Q, L)} = e &\Leftrightarrow d(Q, P)^2 = e^2 d(Q, L)^2 \\
&\Leftrightarrow (x - e)^2 + y^2 = e^2(x - \frac{1}{e})^2 \\
&\Leftrightarrow x^2 - 2ex + e^2 + y^2 = e^2x^2 - 2ex + 1 \\
&\Leftrightarrow (1 - e^2)x^2 + y^2 = 1 - e^2 \\
&\Leftrightarrow x^2 + \frac{y^2}{1 - e^2} = 1.
\end{aligned}$$

To scale back to the original size, we need to stretch the plane by a factor of a. This is done by making the substitutions

$$x \rightsquigarrow \frac{x}{a} \quad \text{and} \quad y \rightsquigarrow \frac{y}{a},$$

which produces the following equation, which is an alternate form of the equation of the locus of the generalized parabola with focus $(ae, 0)$ and directrix $x = \frac{a}{e}$:

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.}$$

This is exactly the equation we had for ellipses and hyperbolas parameterized by a and e!

If $0 < e < 1$, this is the equation of an ellipse; set $b^2 = a^2(1-e^2)$. Then $c^2 = a^2 - b^2 = a^2(1-(1-e^2)) = a^2e^2$, so $c = ae$, and $e = \frac{c}{a}$ is the eccentricity.

If $e > 1$, this is the equation of a hyperbola; set $b^2 = a^2(e^2-1)$, so $c^2 = a^2+b^2 = a^2(e^2-1+1) = a^2e^2$, and again, $e = \frac{c}{a}$.

5.5. **Conclusion.** We have shown how ellipses and hyperbolas are generalizations of parabolas, that their foci act as generalizations of a parabolic focus, that they also have directrices, and that they are parameterized by their eccentricities in this generalization.

6. Rotation

The matrix of rotation by θ is

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

The inverse matrix is

$$R_{=-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Let the (u, v) plane be the (x, y) plane rotated by θ degrees. Then

$$R_\theta \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ y \end{bmatrix} \quad \text{and} \quad R_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}.$$

This produces the equations

$$\begin{aligned} x &= u\cos\theta - v\sin\theta \\ y &= u\sin\theta + v\cos\theta \end{aligned} \tag{1}$$

and

$$\begin{aligned} u &= x\cos\theta + y\sin\theta \\ v &= y\cos\theta - x\sin\theta. \end{aligned} \tag{2}$$

To shorten notation, set $\alpha = \sin\theta$ and $\beta = \cos\theta$. Then we have

$$\boxed{x = \beta u - \alpha v \quad \text{and} \quad y = \alpha u + \beta v.}$$

It is convenient to expand these as follows.

$$\begin{aligned} x^2 &= \beta^2 u^2 - 2\alpha\beta uv + \alpha^2 v^2; \\ y^2 &= \alpha^2 u^2 + 2\alpha\beta uv + \beta^2 v^2; \\ xy &= \alpha\beta u^2 - \alpha\beta v^2 + (\beta^2 - \alpha^2)uv. \end{aligned}$$

Given

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

we may eliminate the B term by a linear change of variable as in equations (1), where

$$\cot 2\theta = \frac{A - C}{B}.$$

Let $a = A - C$ and $b = B$, and set $c = \sqrt{a^2 + b^2}$. Then

$$\sin 2\theta = \frac{b}{c} \quad \text{and} \quad \cos 2\theta = \frac{a}{c}.$$

Using a half-angle formula,

$$\sin\theta = \sqrt{\frac{1 - \frac{a}{c}}{2}} = \sqrt{\frac{c - a}{2c}} \quad \text{and} \quad \cos\theta = \sqrt{\frac{1 + \frac{a}{c}}{2}} = \sqrt{\frac{c + a}{2c}}.$$

Thus

$$\boxed{\alpha^2 = \frac{c - a}{2c} \quad \text{and} \quad \beta^2 = \frac{c + a}{c}.}$$

BASIS Scottsdale
E-mail address: paul.bailey@basised.com